Monday, November 23, 2015

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Problem 3

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

Solution. Compare it to $\sum_{n=1}^{\infty} \frac{1}{2n}$, which we know diverges.

$$\begin{aligned} \frac{1}{2n-1} &\geq \frac{1}{2n}, \\ 2n &\geq 2n-1, \end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges.

Problem 4

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$.

Solution. Compare it to $\sum_{n=1}^{\infty} \frac{1}{3n^2}$, which we know converges.

$$\begin{aligned} \frac{1}{3n^2+2} &\leq \frac{1}{3n^2}, \\ 3n^2 &\leq 3n^2+2, \end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{1}{3n^2+2}$ converges.

Problem 5

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$.

Solution. Compare it to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which we know diverges ($p < 1$).

$$\begin{aligned} \frac{1}{\sqrt{n}-1} &\geq \frac{1}{\sqrt{n}}, \\ \sqrt{n} &\geq \sqrt{n}-1, \end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges.

Problem 6

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n+3}$.

Solution. Compare it to $\sum_{n=1}^{\infty} \frac{4^n}{5^n}$, which we know converges ($|r| < 1$).

$$\begin{aligned} \frac{4^n}{5^n+3} &\leq \frac{4^n}{5^n}, \\ 4^n \cdot 5^n &\leq 4^n(5^n+3), \end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{4^n}{5^n+3}$ converges.

Problem 7

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$.

Solution. We have to be a wee bit creative here. Compare $\sum_{n=2}^{\infty} \frac{\ln n}{n+1}$ to $\sum_{n=1}^{\infty} \frac{1}{n+1}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and we see that

$$\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$$

so it follows that $\sum_{n=1}^{\infty} \frac{1}{n+1}$ diverges. (The first term cannot make the difference between convergence and divergence.)

Now compare the terms.

$$\begin{aligned} \frac{\ln n}{n+1} &\geq \frac{1}{n+1}, \\ \ln n &\geq 1, \end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$ converges.

You could solve this problem by comparing $\sum_{n=1}^{\infty} \frac{\ln n}{n+1}$ directly to $\sum_{n=1}^{\infty} \frac{1}{n}$, but then you would have to verify the inequality $n \ln n \geq n+1$ for all n from some point on. That can be done, but it is harder to do than what we just did.

Problem 8

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$.

Solution. Without the $+1$, this expression would be $\frac{1}{n^{3/2}}$. So let's compare it to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which we know converges.

$$\begin{aligned} \frac{1}{\sqrt[3]{n+1}} &\leq \frac{1}{n^{3/2}}, \\ n^{3/2} &\leq \sqrt[3]{n^3+1}, \\ n^3 &\leq n^3+1, \end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges.

Problem 9

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$.

Solution. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$, which we know converges (geometric).

$$\begin{aligned}
\frac{1}{n!} &\leq \frac{1}{2^{n-1}}, \\
2^{n-1} &\leq n!, \\
2 \cdot 2 \cdot 2 \cdots 2 \ (n-1 \text{ factors}) &\leq n(n-1)(n-2) \cdots 2 \cdot 1 \ (n \text{ factors}), \\
2 \cdot 2 \cdot 2 \cdots 2 \ (n-1 \text{ factors}) &\leq n(n-1)(n-2) \cdots 2 \ (n-1 \text{ factors}),
\end{aligned}$$

This is true because each factor on the left is less than or equal to the corresponding factor on the right. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Problem 10

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$.

Solution. Without the 4 and -1, the denominator would be $n^{1/3}$, so let's compare it to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^{1/3}}$, which we know diverges.

$$\begin{aligned}
\frac{1}{4\sqrt[3]{n}-1} &\geq \frac{1}{4n^{1/3}}, \\
4n^{1/3} &\geq 4\sqrt[3]{n} - 1,
\end{aligned}$$

which is clearly true and all the steps are logically reversible. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ diverges.

Problem 11

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} e^{-n^2}$.

Solution. The series $\sum_{n=1}^{\infty} e^{-n}$ is a convergent geometric series ($r = e^{-n}$), so let's compare $\sum_{n=1}^{\infty} e^{-n^2}$ to $\sum_{n=1}^{\infty} e^{-n}$.

$$\begin{aligned} e^{-n^2} &\leq e^{-n}, \\ e^{n} &\leq e^{n^2}, \\ 1 &\leq e^{n^2-1}, \end{aligned}$$

which is clearly true for all $n \geq 1$ and each step is logically reversible. Therefore, $\sum_{n=1}^{\infty} e^{-n^2}$ converges.

Problem 12

Problem. Use the Direct Comparison Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$.

Solution. Without the -1, this would be $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$, which is a divergent geometric series ($r = \frac{3}{2}$). So let's compare $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$ to $\sum_{n=1}^{\infty} \frac{3^n}{2^n}$.

$$\begin{aligned} \frac{3^n}{2^n - 1} &\geq \frac{3^n}{2^n}, \\ 3^n \cdot 2^n &\geq 3^n(2^n - 1), \\ 2^n &\geq 2^n - 1, \end{aligned}$$

which is clearly true and each step is logically reversible. Therefore, $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$ diverges.